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On the Convergence of Odd-Degree Spline Interpolation

CARL DE BOOR

Division of Mathematical Sciences, Purdue University, Lafayette, Indiana 47907

1. INTRODUCTION

For $k \geqslant 1$, let S_π^k denote the set of polynomial splines of order k (or, degree $k-1$) on the partition $\pi = \{t_i\}_{i=0}^n$ of the unit interval. Here

$$0 = t_0 < t_1 < \ldots < t_n = 1,$$

so that S_π^k consists of all $s \in C^{k-2}$ [0, 1] which on each of the intervals (t_i, t_{i+1}), $i = 0, \ldots, n-1$, reduce to a polynomial of degree $\leqslant k-1$.

For $k = 2m$, let I_π^k denote the linear operation of spline interpolation, i.e., [4], for each $f \in C^{m-1}$ [0, 1], $I_\pi^k f$ is the unique element of S_π^k satisfying

$$\begin{aligned} (I_\pi^k f)(t_i) &= f(t_i), \qquad i = 0, \ldots, n, \\ (I_\pi^k f)^{(j)}(t_i) &= f^{(j)}(t_i), \qquad i = 0, n; j = 1, \ldots, m-1. \end{aligned} \tag{1.1}$$

We are interested in the behavior of

$$\|(f - I_\pi^k f)^{(j)}\|_\infty, \qquad j = 0, \ldots, 2m-1,$$

as the *norm* of π,

$$\|\pi\| = \max(t_{i+1} - t_i),$$

tends to zero. Here, and below,

$$\|g\|_\infty = \sup\{|g(t)| : 0 \leqslant t \leqslant 1\}.$$

Much is known about this problem in certain special cases. For one, the case $k = 4$ of cubic spline interpolation has been covered extensively by many: [1], [2], [3], [12], [14], [15]. For the purposes of this note, Sharma and Meir's result [14] is the most pertinent. They prove that if $f \in C^2$ [0, 1], then

$$\|(f - I_\pi^4 f)^{(2)}\|_\infty \leqslant 4\omega(f^{(2)}; \|\pi\|)$$

for all partitions π of [0, 1], where

$$\omega(g; \delta) = \sup\{|g(s) - g(t)| : |s - t| \leqslant \delta, \quad s, t \in [0, 1]\}$$

is the modulus of continuity of g on [0, 1].

This implies [6] that

$$\|f - I_\pi^4 f\|_\infty \leqslant K\|\pi\|^4$$

for all $f \in C^{(3)}[0, 1]$ with $\omega(f^{(3)}; \delta) \leqslant \delta$, for all $\delta \geqslant 0$, the constant K being independent of π or f. A similar result had been obtained earlier [3] under some restriction on π.

For $k > 4$, little is known except in the case of a uniform partition [1], [9], [13], [16], [17]. There are some results [10], [14] for $k = 6$ in the limiting case that all points of π are repeated twice, i.e., value as well as first derivative are interpolated at each t_i, and, correspondingly, the elements of $S_\pi{}^6$ are merely in $[C^3][0, 1]$.

In this note, it is proved that for all $f \in C^{(3)}[0, 1]$,

$$\|(f - I_\pi{}^6 f)^{(3)}\|_\infty \leqslant K_3\, \omega(f^{(3)}; \|\pi\|),$$

where the constant K_3 does not depend on π or f.

It is hoped that the method of proof will be useful in the treatment of the general case. The analysis is therefore carried through for arbitrary k up to the point where the complexity of certain computations makes me settle for $k = 6$.

2. Least Squares Approximation by Splines

Let $m \geqslant 2$, and let $L_\pi{}^m$ denote the linear projector on $C[0, 1]$ which associates with each $g \in C[0, 1]$ its best approximation $L_\pi{}^m g$ in $S_\pi{}^m$ with respect to the norm

$$\|g\|_2 = \left[\int_0^1 |g(t)|^2\, dt\right]^{1/2}.$$

Lemma 2.1. *If there exists a constant c_m, independent of π, such that*

$$\|L_\pi{}^m\|_\infty = \sup\{\|L_\pi{}^m g\|_\infty / \|g\|_\infty;\ g \in C[0, 1]\} \leqslant c_m,$$

then, for all $f \in C^m[0,1]$,

$$\|(f - I_\pi^{2m})^{(m)}\|_\infty \leqslant K_m\, \omega(f^{(m)}; \|\pi\|),$$

where K_m is independent of π or f.

Proof. By [4], if $f \in C^m[0,1]$, then

$$(I_\pi^{2m} f)^{(m)} = L_\pi{}^m f^{(m)}.$$

Hence, as $L_\pi{}^m$ is a linear projector with $S_\pi{}^m$ as its range,

$$\|(f - I_\pi^{2m} f)^{(m)}\|_\infty \leqslant (1 + \|L_\pi{}^m\|_\infty)\, \mathrm{dist}(f^{(m)}, S_\pi{}^m),$$

where

$$\mathrm{dist}(g, S_\pi{}^m) = \inf_{s \in S_\pi^m} \|g - s\|_\infty.$$

Since, by [5], for $g \in C[0,1]$,

$$\operatorname{dist}(g, S_\pi^m) \leqslant \hat{D}_m \, \omega(g; \|\pi\|),$$

where the constant $\hat{D}_m$ depends neither on g nor on π, the conclusion follows. Q.E.D.

COROLLARY. *Under the assumption of Lemma* 2.1, *there exists a constant* C_m, *independent of* π, *such that for all* $f \in C^{2m-1}[0,1]$ *with* $\omega(f^{(2m-1)}; \delta) \leqslant \delta$, *for all* $\delta \geqslant 0$,

$$\|f - I_\pi^{2m} f\|_\infty \leqslant C_m \|\pi\|^{2m}.$$

Proof. By [5], there exists a constant k_1, independent of f or π, such that

$$\operatorname{dist}(f^{(m)}, S_\pi^m) \leqslant k_1 \|\pi\|^m$$

for all f satisfying the above assumptions. Hence,

$$\|(f - I_\pi^{2m})^{(m)}\|_\infty \leqslant (1 + c_m) \, k_1 \|\pi\|^m$$

follows. But as $I_\pi^{2m} f$ interpolates f at the points of π, repeated application of Rolle's Theorem yields from this

$$\|(f - I_\pi^{2m})^{(j)}\|_\infty \leqslant (1 + c_m) \, k_1 \, p_j \|\pi\|^{2m-j}, \qquad j = 0, \ldots, m,$$

where, again, the constants p_j do not depend on f or π. Q.E.D.

For the remainder of this section, we shall be concerned with bounding $\|L_\pi^m\|_\infty$.

First, a general observation. If $\{x_i\}_{i=1}^r$ is a sequence of points in a real normed linear space X, and $\{\lambda_i\}_{i=1}^r$ is a sequence of continuous linear functionals on X, then the conditions

$$Pf = \sum_{i=1}^r \alpha_i x_i, \qquad \lambda_i(f - Pf) = 0, \qquad i = 1, \ldots, r, \quad \text{for all } f \in X,$$

define a continuous linear projector P on X, with range the linear span of $\{x_i\}_1^r$, provided the matrix

$$A = (\lambda_i x_j)_{i,j=1}^r$$

is nonsingular. We shall refer to P in this case as being *given* or *defined by* $\{x_i\}_1^r$ *and* $\{\lambda_i\}_1^r$.

LEMMA 2.2. *Let X be a real normed linear space and let P be the linear projector on X given by* $\{x_i\}_1^r \subset X$ *and* $\{\lambda_i\}_1^r \subset X^*$. *Then*

$$\|P\| \leqslant c \|A^{-1}\|_\infty \cdot \max_i \|\lambda_i\|, \tag{2.1}$$

where

$$c = \sup_{\alpha \in \mathbf{R}^r} \left\| \sum_{i=1}^{r} \alpha_i x_i \right\| / \|\alpha\|_\infty .$$

Remark. We use the notations

$$\|\alpha\|_\infty = \max_i |\alpha_i|, \quad \text{for all } \alpha = (\alpha_i) \in \mathbf{R}^r,$$

and

$$\|B\|_\infty = \sup\{\|B\alpha\|_\infty / \|\alpha\|_\infty : \alpha \in \mathbf{R}^r\},$$

where B is any real $r \times r$ matrix.

Proof of Lemma 2.2. Let $f \in X$ and $Pf = \sum_{i=1}^r \alpha_i x_i$. Then

$$\|Pf\| \leqslant c\|\alpha\|_\infty, \quad \text{and} \quad A\alpha = (\lambda_i f)_{i=1}^r .$$

Hence

$$\|Pf\| \leqslant c\|A^{-1}\|_\infty \cdot \|(\lambda_i f)\|_\infty \leqslant c\|A^{-1}\|_\infty \cdot \max_i \|\lambda_i\| \cdot \|f\|,$$

which proves (2.1), as f was arbitrary. Q.E.D.

As is well known, L_π^m is given by $\{x_i\}_1^r$ and $\{\lambda_i\}_1^r$ where $\{x_i\}_1^r$ is any basis of S_π^m, and

$$\lambda_i f = \int_0^1 y_i(t) f(t)\, dt, \qquad i = 1, \ldots, r, \quad \text{for all } f \in C[0,1],$$

with $\{y_i\}_1^r$ any basis of S_π^m. We shall choose x_i and y_i in such a way that

$$\sup_{\alpha \in \mathbf{R}^r} \|\alpha_i x_i\|_\infty / \|\alpha\|_\infty = \max_i \|\lambda_i\| = 1.$$

For then, by Lemma 2.2,

$$\|L_\pi^m\|_\infty \leqslant \|A^{-1}\|_\infty,$$

and the problem of bounding L_π^m reduces to bounding the matrix $A = (\lambda_i x_j)$ below in the uniform norm, uniformly with respect to π.

For ease of notation, it is convenient to extend the partition π of [0, 1] by the adjunction of points

$$t_{1-2m} < \ldots < t_{-1} < 0, \qquad 1 < t_{n+1} < \ldots < t_{n+2m-1},$$

which, for the present, are otherwise arbitrary. Later, the first few of the additional t_i's will be made to coalesce, i.e.,

$$t_{1-m} = \ldots = t_{-1} = 0, \qquad 1 = t_{n+1} = \ldots = t_{n+1-m}. \tag{2.2}$$

Define

$$x_i(t) = g(t_i, \ldots, t_{i+m}; t)(t_{i+m} - t_i), \quad \text{for all } t \in \mathbf{R}, \tag{2.3}$$

where $g(t_i, \ldots, t_{i+m}; t)$ is the mth divided difference in s, on the points $t_i, \ldots, t_{i+m}$, of the function

$$g(s; t) = (s - t)_+^{m-1}. \tag{2.4}$$

Further, set

$$\lambda_i f = m \int_{-\infty}^{\infty} g(t_i, \ldots, t_{i+m}; t) f(t)\, dt. \tag{2.5}$$

The following facts about x_i and λ_i are known [8], [5];

LEMMA 2.3 (i) *The function $x_i(t)$ vanishes outside the interval $[t_i, t_{i+m}]$ and is positive on (t_i, t_{i+m}).*

(ii) *The sequence of functions $\{x_i\}_{i=1-m}^{n-1}$ (restricted to the interval* $[0,1]$) *is a basis for S_π^m; further, for all $\alpha_i \in \mathbf{R}$, $i = 1 - m, \ldots, n - 1$, one has*

$$\| \sum_i \alpha_i x_i \|_\infty \leqslant \max_i |\alpha_i|.$$

(iii) *If $f \in C[I]$, with $[t_i, t_{i+m}] \subset I$, then*

$$|\lambda_i f| \leqslant \sup_{t \in I} |f(t)|.$$

COROLLARY. *The linear projector L_π^m is given by $\{x_i\}_{i=1-m}^{n-1}$ and $\{\lambda_i\}_{i=1-m}^{n-1}$ provided* (2.2) *holds. In that case*

$$\|L_\pi^m\|_\infty \leqslant \|A^{-1}\|_\infty, \qquad \text{where } A = (\lambda_i x_j).$$

The calculation of bounds on $\|A^{-1}\|_\infty$ for a given real matrix A is in general difficult. The best-known result concerns strictly diagonally dominant A: If $A = (\alpha_{ij})$, and

$$\min_i \left| \alpha_{ii} - \sum_{j \neq i} |\alpha_{ij}| \right| \geqslant d^{-1} > 0,$$

then A^{-1} exists and

$$\|A^{-1}\|_\infty \leqslant d.$$

This result is applicable to the matrix A under discussion only in the simplest case, $m = 1$.

LEMMA 2.4. *If all $(n-1)$-minors of the $n \times n$ matrix $A = (\alpha_{ij})$ are nonnegative and, for some $\gamma = (\gamma_i)$,*

$$\min_i \left(\sum_{j=1}^{n} (-1)^{i-j} \gamma_j \alpha_{ij} \right) \geqslant d^{-1} > 0,$$

then A^{-1} exists and

$$\|A^{-1}\| \leqslant d\|\gamma\|.$$

Proof. Let B be the algebraic adjoint of A and let D be the diagonal matrix $((-1)^i \delta_{ij})$, where δ_{ij} is the Kronecker delta. Then, by assumption, DBD^{-1} has all entries nonnegative, and $\hat{\gamma} = DAD^{-1}\gamma$ has all components $\geqslant d^{-1} > 0$. Hence

$$\det(A)\gamma = DBD^{-1}(DAD^{-1})\gamma$$

is not zero, unless $B = 0$, which would imply $A = 0$, a contradiction. Therefore A^{-1} exists and $(\hat{\alpha}_{ij}) = DA^{-1}D^{-1}$ has all entries nonnegative. With this,

$$\|\gamma\|_\infty = \|(DA^{-1}D^{-1})\hat{\gamma}\|_\infty = \max_i \left| \sum_{j=1}^{n} \hat{\alpha}_{ij} \hat{\gamma}_j \right|$$

$$\geqslant \left(\max_i \sum_{j=1}^{n} \hat{\alpha}_{ij} \right) \min_i \hat{\gamma}_j \geqslant \|DA^{-1}D^{-1}\|_\infty d^{-1};$$

hence, $\|A^{-1}\|_\infty = \|DA^{-1}D^{-1}\|_\infty \leqslant d\|\gamma\|_\infty$. Q.E.D.

As we shall show in a moment, the matrix $A = (\lambda_i x_j)$ has all minors nonnegative, so that Lemma 2.4 applies. Further, by definition (2.3) of x_j and (2.5) of λ_i,

$$\lambda_i x_j = m(t_{j+m} - t_j) \int_{t_i}^{t_{i+m}} g(t_i, \ldots, t_{i+m}; t) g(t_j, \ldots, t_{j+m}; t) dt,$$

and, therefore, by Lemma 2.3 (i),

$$\lambda_i x_j = 0 \quad \text{if} \quad t_{i+m} \leqslant t_j \quad \text{or} \quad t_{j+m} \leqslant t_i. \tag{2.7}$$

This implies that A is a band matrix, and that

$$\lambda_i x_j = f_{i-j}(t_{i-m+1}, \ldots, t_{i+2m-1}), \qquad i,j = -m+1, \ldots, n-1,$$

where

$$f_r(s_{-m+1}, \ldots, s_{2m-1}) = \begin{cases} m(s_{r+m} - s_r) \int_{s_0}^{s_m} g(s_0, \ldots, s_m; t) g(s_r, \ldots, s_{r+m}; t)\, dt, \\ \quad \text{for } |r| \leqslant m-1, \\ 0 \quad \text{otherwise.} \end{cases}$$

Also, if $\gamma_{-2m+2}, \ldots, \gamma_{n+m-2}$ are any scalars, and (2.2) holds, then

$$\sum_{j=-m+1}^{n-1} \lambda_i(\gamma_j x_j) = \sum_{j=i-m+1}^{i+m-1} \lambda_i(\gamma_j x_j), \qquad i = -m+1, \ldots, n-1, \tag{2.9}$$

since by (2.2) and (2.7),

$$\lambda_i x_j = 0 \quad \text{for} \quad j < -m+1 \quad \text{and} \quad j > n+m-2.$$

Therefore, if $\gamma_j = C(t_j, \ldots, t_{j+m})$, for all j where C is some function of $m+1$ variables, then

$$\sum_{j=-m+1}^{n-1} (-1)^{i-j} \lambda_i(\gamma_j x_j) = \sum_{r=-m+1}^{m-1} (-1)^r C(t_{i+r}, \ldots, t_{i+r+m}) f_r(t_{i-m+1}, \ldots, t_{i+2m-1})$$
$$= F(t_{i-m+1}, \ldots, t_{i+2m-1}), \qquad i = -m+1, \ldots, n-1. \tag{2.10}$$

With this, Lemma 2.4 shows that bounding $\|A^{-1}\|_\infty$ independently of π reduces to showing that for some choice of the function C in (2.10), with

$$|C(s_0, \ldots, s_m)| \leqslant 1 \quad \text{whenever} \quad s_0 \leqslant \ldots \leqslant s_m;\ s_0 < s_m,$$

the function F defined by (2.10) satisfies

$$F(s_{m+1}, \ldots, s_{2m-1}) \geqslant d^{-1} > 0,$$

whenever $s_{-m+1} \leqslant \ldots \leqslant s_{2m-1};\ s_i < s_{i+m}$, for all i.

Theorem 2.1. *Let $C(s_0, \ldots, s_m)$ be a real-valued function defined on*

$$T = \{(s_i)_{i=0}^m \in \mathbf{R}^{m+1} : s_0 \leqslant s_1 \leqslant \ldots \leqslant s_m;\ s_0 < s_m\}$$

and continuous there, which satisfies

$$\sup_T |C(s_0, \ldots, s_m)| \leqslant 1.$$

Further, define F on

$$\hat{T} = \{(s_i)_{i=-m+1}^{2m-1} \in \mathbf{R}^{3m-1} : s_{-m+1} \leqslant \ldots \leqslant s_{2m-1};\ s_j < s_{j+m} \text{ for all } j\}$$

by

$$F(s_{-m+1}, \ldots, s_{2m-1}) = \sum_{j=-m+1}^{m-1} (-1)^j C(s_j, \ldots, s_{j+m})\, a_j, \tag{2.11}$$

where

$$a_j = m(s_{j+m} - s_j) \int_{s_0}^{s_m} g(s_0, \ldots, s_m; t)\, g(s_j, \ldots, s_{j+m}; t)\, dt,$$
$$j = -m+1, \ldots, n-1. \tag{2.12}$$

If

$$\inf_T F(s_{-m+1}, \ldots, s_{2m-1}) \geqslant d_m^{-1} > 0,$$

then

$$\|L_\pi{}^m\|_\infty \leqslant d_m, \qquad \textit{for all partitions } \pi. \tag{2.13}$$

Proof. By the corollary to Lemma 2.3, it is sufficient to prove that $\|A^{-1}\|_\infty \leqslant d_m$ with $A = (\lambda_i x_j)$.

It follows from ([*11*]; Ch. 10, Theorem 4.1) or ([7]; Ch. III, Section 2 (3)) that all minors of the matrix

$$(g(s_i, \ldots, s_{i+m}; u_j))_{i,j=1}^r$$

are nonnegative, provided

$$s_1 \leqslant s_2 \leqslant \ldots \leqslant s_r;\ s_j < s_{j+m} \quad \text{for all } j;\ u_1 < u_2 < \ldots < u_r,$$

and $r \geqslant 1$. Since, with the condition (2.2),

$$\lambda_i x_j = m(t_{j+m} - t_j) \int_0^1 g(t_i, \ldots, t_{i+m}; t) g(t_j, \ldots, t_{j+m}; t)\, dt,$$
$$i,j = -1 + m, \ldots, n-1,$$

the "basic composition formula" ([*11*]; pp. 16–17) implies that all minors of the matrix A are nonnegative.[1] This, together with the discussion preceding the theorem, concludes the proof. Q.E.D.

Remark. Since the function F defined by (2.11) and (2.12) is continuous on $\hat{T}$, it is sufficient to show that

$$F(s_{-m+1}, \ldots, s_{2m-1}) \geqslant d_m^{-1}$$

for all $s_{-m+1} < s_{-m+2} < \ldots < s_{2m-1}$, to prove (2.13).

3. Quintic Spline Interpolation

The simplest case covered by the analysis of the preceding section is that of cubic spline interpolation, i.e., $k = 4$ or $m = 2$. In this case, the a_j's of (2.12) are given by

$$a_j = \tfrac{1}{3} \cdot \begin{cases} (s_1 - s_0)/(s_2 - s_0), & j = -1, \\ 2, & j = 0, \\ (s_2 - s_1)/(s_2 - s_0), & j = 1. \end{cases}$$

[1] The author gratefully acknowledges that this argument was pointed out to him by W. Studden.

Hence, with $C(s_0, s_1, s_2) \equiv 1$, one gets

$$F(s_{-1}, \ldots, s_3) = \frac{1}{3}\left\{-\frac{s_1 - s_0}{s_2 - s_0} + 2 - \frac{s_2 - s_1}{s_2 - s_0}\right\} = \tfrac{1}{3}.$$

Therefore, $\|L_n{}^2\|_\infty \leqslant 3$.

The next simplest case is quintic spline interpolation, i.e., $k = 6$ or $m = 3$. In this case

$$a_j = \tfrac{1}{10} \cdot \begin{cases} \beta_{-1}\dfrac{s_1 - s_0}{s_3 - s_0}, & j = -2, \\[2ex] \beta_0 - a_{-2} + 2\dfrac{s_1 + s_2 - 2s_0}{s_3 - s_0}, & j = -1, \\[2ex] 2(3 - \beta_0), & j = 0, \\[2ex] \beta_0 - a_2 + 2\dfrac{2s_3 - s_2 - s_1}{s_3 - s_0}, & j = 1, \\[2ex] \beta_1\dfrac{s_3 - s_2}{s_3 - s_0}, & j = 2, \end{cases}$$

where

$$\beta_j = \frac{(s_{j+2} - s_{j+1})^2}{(s_{j+3} - s_{j+1})(s_{j+2} - s_j)}, \quad \text{for all } j. \tag{3.1}$$

One computes

$$10 \sum_{j=-2}^{2} (-1)^j a_j = 2 - 4\beta_0 + 2[\beta_{-1}(s_1 - s_0) + \beta_1(s_3 - s_2)]/(s_3 - s_0). \tag{3.2}$$

Hence, as

$$0 \leqslant \beta_j \leqslant 1, \quad \text{for all } j,$$

the choice $C(s_0, \ldots, s_3) \equiv 1$ will not give the desired result. We shall now show that with

$$C(s_0, \ldots, s_3) = \tfrac{1}{2}(1 + \beta_0),$$

one gets

$$F(s_{-2}, \ldots, s_5) \geqslant 1/30.$$

One finds that

$$\begin{aligned} 10 \sum_{j=-2}^{2} (-1)^j \beta_j a_j = {} & 6\beta_0 - \beta_0(2\beta_0 + \beta_{-1} + \beta_1) \\ & - 2[\beta_{-1}(s_1 - s_0) + \beta_1(s_3 - s_2)]/(s_3 - s_0) \\ & - \beta_{-1}[2(s_2 - s_0) - \beta_{-1}(s_1 - s_0)]/(s_3 - s_0) \\ & - \beta_1[2(s_3 - s_1) - \beta_1(s_3 - s_2)]/(s_3 - s_0) \\ & + \beta_{-1}\beta_{-2}(s_1 - s_0)/(s_3 - s_0) + \beta_1\beta_2(s_3 - s_2)/(s_3 - s_0). \end{aligned} \tag{3.3}$$

Hence, by omitting the last two terms in (3.3) (which are nonnegative) and combining (3.2) with (3.3), one gets

$$\begin{aligned} 20F(s_{-2}, \ldots, s_5) \geqslant 2 &+ \beta_0(2 - 2\beta_0 - \beta_{-1} - \beta_1) \\ &+ (s_3 - s_0)^{-1} [\beta_{-1}(\beta_{-1}(s_1 - s_0) - 2(s_2 - s_0)) \\ &+ \beta_1(\beta_1(s_3 - s_2) - 2(s_3 - s_1))]. \end{aligned} \tag{3.4}$$

Now,

$$\beta_1 = \frac{(s_3 - s_2)^2}{(s_3 - s_1)(s_4 - s_2)} \leqslant \frac{s_3 - s_2}{s_3 - s_1} = 1 - \frac{s_2 - s_1}{s_3 - s_1},$$

hence

$$-\beta_0 - \beta_1 \geqslant -\frac{s_2 - s_1}{s_3 - s_1} - \left(1 - \frac{s_2 - s_1}{s_3 - s_1}\right) = -1.$$

Similarly,

$$-\beta_0 - \beta_{-1} \geqslant -1.$$

Therefore,

$$\beta_0(2 - 2\beta_0 - \beta_1 - \beta_{-1}) \geqslant 0. \tag{3.5}$$

Next,

$$h(s) = s[s(s_1 - s_0) - 2(s_2 - s_0)]$$

has negative slope on $0 < s < 1$. Hence, since

$$0 \leqslant \beta_{-1} \leqslant \frac{s_1 - s_0}{s_2 - s_0} \leqslant 1,$$

one has

$$\begin{aligned} \beta_{-1}[\beta_{-1}(s_1 - s_0) - 2(s_2 - s_0)] &\geqslant \frac{s_1 - s_0}{s_2 - s_0}\left[\frac{(s_1 - s_0)^2}{s_2 - s_0} - 2(s_2 - s_0)\right] \\ &= \frac{(s_1 - s_0)^3}{(s_2 - s_0)^2} - 2(s_1 - s_0). \end{aligned} \tag{3.6}$$

Combining (3.4), (3.5) and (3.6) (with an analogous estimate for the term in (3.4) involving β_1), one gets

$$20F(s_{-2}, \ldots, s_5) \geqslant \left[\frac{(s_1 - s_0)^3}{(s_2 - s_0)^2} + \frac{(s_3 - s_2)^3}{(s_3 - s_1)^2} + 2(s_2 - s_1)\right] / (s_3 - s_0). \tag{3.7}$$

Now set

$$s_1 - s_0 = a, \qquad s_2 - s_1 = b, \qquad s_3 - s_2 = c,$$

to simplify notation. Then by (3.7),

$$\begin{aligned} 20F(s_{-2}, \ldots, s_5) \geqslant [(a^3 + b(a + b)^2)(b + c)^2 + (c^3 + b(c + b)^2)(a + b)^2]/ \\ /(a + b + c)(a + b)^2(c + b)^2. \end{aligned} \tag{3.8}$$

One has

$$\tfrac{2}{3}(a+\tfrac{1}{2}b)(a+b)^2 = \tfrac{2}{3}a^3 + \tfrac{5}{3}a^2b + \tfrac{4}{3}ab^2 + \tfrac{1}{3}b^3;$$

hence,

$$\begin{aligned} a^3 + b(a+b)^2 &= a^3 + a^2b + 2ab^2 + b^3 \\ &= \tfrac{2}{3}(a+\tfrac{1}{2}b)(a+b)^2 + \tfrac{1}{3}(a(a-b)^2 + ab^2 + 2b^3) \\ &\geqslant \tfrac{2}{3}(a+\tfrac{1}{2}b)(a+b)^2, \end{aligned}$$

since a, b, and c are all nonnegative. Therefore,

$$\begin{aligned} 20F(s_{-2},\ldots,s_5) &\geqslant \tfrac{2}{3}[(a+\tfrac{1}{2}b)(a+b)^2 + (c+\tfrac{1}{2}b)(c+b)^2(a+b)^2]/ \\ &\quad /(a+b+c)(a+b)^2(c+b)^2 \\ &= \tfrac{2}{3}. \end{aligned} \tag{3.9}$$

Because of Theorem 2.1 and its corollary, this proves

Theorem 3.1. *For all partitions π of* [0, 1], *the linear projector L_π^3 on C* [0, 1] *of least-squares approximation by S_π^3 is bounded in the uniform norm, independently of π. One has the estimate*

$$\|L_\pi^3\|_\infty \leqslant 30.$$

Hence, there exists a constant K_6 such that for all partitions π of [0, 1] *and all $f \in C^5$* [0, 1] *with $\omega(f^{(5)}, \delta) \leqslant \delta$, for all $\delta \geqslant 0$,*

$$\|f^{(j)} - (I_\pi^6 f)^{(j)}\|_\infty \leqslant K_6 \|\pi\|^{6-j}, \qquad j = 0, \ldots, 3,$$

where I_π^6 denotes interpolation by quintic splines as defined in (1.1).

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